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Uniqueness results for the identification of a time-dependent diffusion coefficient

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Abstract. This paper deals with the problem of determining the time dependent thermal diffusivity coefficient of a medium when the evolution of the temperature in a part of it is known. Such situations arise in contexts of food technology, when used thermal processes at high pressures for extending the shelf life of the food, in order to preserve its nutritional and organoleptic properties (see [5] and [13]).

The phenomenon is modeled by the heat equation involving a term which depends on the source temperature and pressure increase, and appropriate initial and boundary conditions.

We study the inverse problem of determining time dependent thermal diffusivities, when some temperature measurements at the border and inside the medium are known. We prove the uniqueness of the inverse problem solution under suitable a priori assumptions on regularity, size and growth of k.

1. Introduction

Physical processes that combine high pressure with moderate temperatures can be modeled (see, for example, [5] and [13]) by equations with different physical parameters whose values, although often known for atmospheric pressure, are to be determined for arbitrary values of pressure. In this work thermal conductivity will be assumed to depend only on pressure, $k = k(P)$. This hypothesis is suitable, for example, in processes in which the temperature range is moderate and there is no phase change. We study the problem of identifying $k(P)$ from certain experimental measurements of the temperature. These measurements correspond to an experiment in which the pressure curve P is an strictly increasing function. Therefore, the problem of identifying $k(P)$ is equivalent to identify $k(t) = k(P(t))$. The aim is to determine suitable conditions and measurements under which the uniqueness of function k can be guaranteed.

When a cylindrical domain Ω is considered, under suitable conditions (see [15]) a one–dimensional radial model can be a good approximation of the model. Then, the model can be written in cartesian coordinates (for convenience in this paper) as

$$\begin{cases} \dfrac{\partial T}{\partial t} - k(t)\Delta T = \alpha P'(t) T & \text{in } B_R \times (0, t_{\mathrm{f}}) \\ k(t)\dfrac{\partial T}{\partial \mathbf{n}} = h\,(T^e(t) - T) & \text{on } \partial B_R \times (0, t_{\mathrm{f}}) \\ T = T_0 & \text{on } B_R \times \{0\}, \end{cases} \tag{1}$$

where $R > 0$, $t_{\mathrm{f}} > 0$, $B_R(0) \subset R^2$ is a disk with radius R and centre $(0,0)$, ∂B_R is its boundary. Furthermore, $\alpha = \dfrac{\hat{\alpha}}{\varrho C_p}$ with $\hat{\alpha} > 0$ is the coefficient of thermal expansion, $\rho > 0$ the density and $C_p > 0$ the specific heat; $P \in \mathcal{C}^1([0, t_{\mathrm{f}}])$ is the pressure at time t; $k(t) \geq k_0 > 0$ is the thermal diffusivity; T^e is the external temperature; $\mathbf{n}$ is the outward unit normal vector at the boundary of B_R; $h > 0$ is a heat exchange coefficient and T_0 is the initial temperature (assumed to be constant, for the sake of simplicity). Details about this kind of models and the units of all the parameters and functions can be seen in [5] and [13].

The main goal is to know in how many points do we need to know the temperature in order to get a unique possible function $k(t)$, when we only know, a priori, the value of $k(0)$, which correspond to the thermal diffusivity at atmospheric pressure. Furthermore, it is also of interest to know if there is a mathematical way of choosing the best points to make the temperature measurements.

Previous works have dealt with the problem of diffusion coefficient identification in parabolic equations but, to the best of our knowledge, only a few of them consider the case of time dependent coefficients and all the others study the case of space dependent coefficients.

Regarding the identification of time dependent diffusion coefficients, in [7] the author studies the influence of the lower–order terms in a one dimensional parabolic equation on the possibility of unique determination of the time–depending leading coefficient, using the initial condition, the Dirichlet boundary condition and the flux in one end of the interval defining the space domain.

In [8] a similar problem for the one dimensional heat equation is studied, when there is no lower–order terms and the degeneration condition $k(0) = 0$ is satisfied. As in the previous case, the initial condition, the Dirichlet boundary condition and the flux in one end of the interval defining the space domain are supposed to be known.

In [6] the authors study the identification of the thermal diffusivity $k(t)$ for the one dimensional heat equation in the case of non local boundary and integral overdetermination conditions. Under some assumptions on the data they provide a theorem about the existence and uniqueness for the identification of $k(t)$ and propose a numerical method for its computation, when data are given without error.

Regarding the inverse problem of the identification of space dependent diffusion coefficients, in [9] the author study the unique solvability of the inverse problem in a one dimensional non homogeneous parabolic equation under homogeneous Dirichlet and initial conditions and an additional condition of integral overdetermination with respect to time.

In Chapter 3 of [2] the leading coefficient of the one dimensional heat equation appears in divergence form and the corresponding identification problem is solved by using the quasi–inversion method, assuming that the initial condition and Dirichlet boundary conditions are known, together with the flux in one of the ends of the space interval, for a certain time interval.

In [4] authors use the enclosure method (see [3]) to study a problem of reconstruction of inclusions (parts of the spatial domain with conductivity values different from a known reference value) in a three dimensional heat conductive body, when the temperature and the heat flux are known on the boundary for a finite time interval. They determine the minimum radius of the open ball centered at a given point that contains the inclusions.

One can also find in the literature the solution of the problem of identification of an space dependent lower–term coefficient (see [11]) or a time dependent coefficient multiplying the time derivative of the solution of the direct problem (see [10]).

The main result of this work is Theorem 20, which shows that, under suitable a priori assumptions on regularity, size and growth of $k(t)$, the inverse problem of the identification of $k(t)$ in boundary value problem (1) has a unique solution, when the value of the temperature is known at $r = R$ and $r = r_0$, with $0 \leq r_0 < R$. The proof of this theorem uses the result in Theorem 12, which provides an integral expression for the temperature at an arbitrary radius r in terms of the temperature at radius R. Furthermore, it is shown that, together with $r = R$, the central point $r_0 = 0$ is the best point to do the measurements in order to use them to identify $k(t)$, since the a priori estimates needed for this case are less restrictive.

2. Direct problem. Qualitative Analysis

Let $X = \left\{\varphi \in \mathcal{C}^{2,1}(B_R \times (0, t_\mathrm{f})) \cap \mathcal{C}^{1,0}(\overline{B_R} \times [0, t_\mathrm{f}])\right\}$, where $C^{a,b}(A, B)$ denotes the set of functions $\varphi : A \times B \to \mathbb{R}$ with a continuous derivatives in A and b continuous derivatives in B. The following result holds

Theorem 1 *Suppose that k and T^e are Lipschitz continuous functions in $[0, t_\mathrm{f}]$, P' is Hölder continuous in $[0, t_\mathrm{f}]$ and compatibility condition*

$$T^e(0) = T_0$$

holds. Then Problem (1) has a unique (classical) solution $T \in X$. Moreover, T is a radial function.

Proof. The existence, uniqueness and regularity of solution can be seen in [12, Theorem 5.18]. It is straightforward to show that the solution is radial. □

By means of the changes of variable

$$u(x,t) = T(x,t)e^{\alpha(P(0)-P(t))} \tag{2}$$

and

$$v(x,t) = u(x,t) - T_0, \tag{3}$$

we can rewrite Problem (1) as

$$\begin{cases} \dfrac{\partial v}{\partial t} - k(t)\Delta v = 0 & \text{in } \ B_R \times (0, t_\mathrm{f}) \\ k(t)\dfrac{\partial v}{\partial \mathbf{n}} = h\left(f(t) - T_0 - v\right) & \text{on } \partial B_R \times (0, t_\mathrm{f}) \\ v = 0 & \text{on } \ B_R \times \{0\}, \end{cases} \tag{4}$$

where

$$f(t) = T^e(t)\, e^{\alpha(P(0)-P(t))}.$$

Remark 2 The solution v of Problem (4) lies in X and is a radial function. Nevertheless, we still work with cartesian coordinates, because we are going to express the solution as the composition of certain functions with translations. Since translations are not radial functions, they have not a simple representation in polar coordinates.

The *formal adjoint* of the operator used in (4)

$$\mathcal{L}(\phi) = \frac{\partial \phi}{\partial t} - k(t)\Delta\phi,$$

is given by (see, e. g., [16, pág. 170])

$$\mathcal{L}^*(\phi) = -\frac{\partial \phi}{\partial t} - k(t)\Delta\phi.$$

Both operators are related by means of the following easy–to–prove result:

Proposition 3 (Lagrange Identity) *For every $\phi, \psi \in X$ we have*

$$\int_0^{t_\mathrm{f}} \int_{B_R} (\mathcal{L}(\phi)\psi - \phi\mathcal{L}^*(\psi))dxdt = \varrho C_p \int_{B_R} (\phi(x,t_\mathrm{f})\psi(x,t_\mathrm{f}) - \phi(x,0)\psi(x,0))\,dx$$
$$- \int_0^{t_\mathrm{f}} k(t) \int_{\partial B_R} \left(\frac{\partial\phi}{\partial\mathbf{n}}\psi - \phi\frac{\partial\psi}{\partial\mathbf{n}}\right) dxdt.$$

Notation For any function $v \in X$ we denote by $\overline{v} : [0,R] \times [0, 2\pi[\times[0, t_\mathrm{f}] \to \mathbb{R}$ the function defined as

$$\overline{v}(r,\theta,t) = v(r\cos\theta, r\sin\theta, t).$$

If v is radial we write $\overline{v}(r,t)$.

Corollary 4 *If $v \in X$ is the solution of Problem (4), for every $w \in X$ satisfying the final condition*

$$w(x,t_\mathrm{f}) = 0,\ x \in B_R, \tag{5}$$

we have

$$\int_0^{t_\mathrm{f}} \int_{B_R} \mathcal{L}^*(w)\, v dxdt = h \int_0^{t_\mathrm{f}} (f(t) - \overline{v}(R,t) - T_0) \int_{\partial B_R} w\, dxdt - \int_0^{t_\mathrm{f}} k(t)\overline{v}(R,t) \int_{\partial B_R} \frac{\partial w}{\partial\mathbf{n}}\, dxdt. \tag{6}$$

Proof. It suffices to apply Propositión 3, by taking $\phi = v$, the solution of Problem (4) and $\psi = w \in X$ satisfying (5). □

Next proposition provides a comparison principle for Problem (4).

Proposition 5 (Comparison Principle) *Let $v_1, v_2 \in X$ satisfying*

$$\begin{cases} \mathcal{L}(v_1) \le \mathcal{L}(v_2) & \textit{in } B_R \times (0,t_\mathrm{f}) \\ k(t)\dfrac{\partial v_1}{\partial\mathbf{n}} + hv_1 \le k(t)\dfrac{\partial v_2}{\partial\mathbf{n}} + hv_2 & \textit{on } \partial B_R \times (0,t_\mathrm{f}) \\ v_1 \le v_2 & \textit{on } B_R \times \{0\}. \end{cases}$$

Then

$$v_1(x,t) \le v_2(x,t),\ (x,t) \in \overline{B_R} \times [0,t_\mathrm{f}].$$

Proof. Function $v_0 = v_1 - v_2$ satisfies

$$\begin{cases} \mathcal{L}(v_0) \le 0 & \text{in } B_R \times (0,t_\mathrm{f}) \\ k(t)\dfrac{\partial v_0}{\partial\mathbf{n}} + hv_0 \le 0 & \text{on } \partial B_R \times (0,t_\mathrm{f}) \\ v_0 \le 0 & \text{on } B_R \times \{0\}. \end{cases}$$

Applying the Strong Maximum Principle (cf. [14, Theorem 7]), if function v_0 attains a positive maximum at a point Q then $Q \in \partial B_R \times (0,t_\mathrm{f})$ and

$$\frac{\partial v_0}{\partial\mathbf{n}}(Q) > 0,$$

which leads to the contradiction

$$k(t)\frac{\partial v_0}{\partial\mathbf{n}}(Q) + hv_0(Q) > 0.$$

Consequently $v_0 \le 0$ in $\overline{B_R} \times [0,t_\mathrm{f}]$. □

3. Expression of the solution in terms of its values on the boundary

In this section we find an integral representation of the solution of Problem (4) in terms of its values on ∂B_R (these values will be supposed to be known data, obtained by experimental measurements). As a first step, we calculate the fundamental solution of operator $\mathcal{L}^*$ (in the sense of Proposition 7). In order to find it, we use the well–known expression of the fundamental solution of operator $\mathcal{L}$ for the case $k(t) \equiv 1$

$$\xi(x,t) = \frac{H(t)}{4\pi t}\, e^{-\frac{|x|^2}{4t}}, \tag{7}$$

where H is the Heaviside function

$$H(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leq 0. \end{cases}$$

Given $y \in B_R$ and $\tau \in (0, t_{\mathrm{f}})$, we define the function $w : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ given by

$$w(x,t;y,\tau) = \xi(x-y, K(t) - K(\tau)), \tag{8}$$

being

$$K(s) = \int_s^{t_{\mathrm{f}}} k(z)dz,$$

where function k is supposed to be continuously extended by a constant to the whole $\mathbb{R}$.

Lemma 6 *For every $y \in B_R$ and $\tau \in (0, t_{\mathrm{f}})$, functions ξ and w defined by (7) and (8), respectively, satisfy:*

a) $\xi \in \mathcal{C}^\infty((\mathbb{R}^2 \times \mathbb{R})\backslash\{(0,0)\})$. *Thus,* $w \in \mathcal{C}^\infty((\mathbb{R}^2 \times \mathbb{R})\backslash\{(y,\tau)\})$.

b) $\displaystyle\int_{\mathbb{R}^2} \xi(x-y,s)\, dx = H(s)$ *for all* $s \in \mathbb{R}$.

c) Function w is locally integrable in $\mathbb{R}^2 \times \mathbb{R}$, i. e., $w \in L^1_{\mathrm{loc}}(\mathbb{R}^2 \times \mathbb{R})$.

d) $\xi(x-y,s) \to \delta(x-y)$ as $s \to 0^+$ in $\mathcal{D}'(\mathbb{R}^2)$ (the space of distributions on $\mathbb{R}^2$).

e) For each $x \in \mathbb{R}^2$ and $t \in \mathbb{R}$ with $t < \tau$ we have $\mathcal{L}^*(w(x,t;y,\tau)) = 0$.

Proof.

a) Obviously, ξ is a $\mathcal{C}^\infty$ function at least on the whole space $\mathbb{R}^3$ except in the plane $t = 0$. Moreover, ξ is a continuous function in $(x,0)$ for $x \neq 0$, since

$$0 \leq \xi(x_n, t_n) \leq \frac{1}{4\pi|t_n|}\, e^{-\frac{(|x|-1)^2}{|t_n|}} \to 0 \;\text{ as }\; (x_n, t_n) \to (x,0).$$

Similar arguments lead to prove the same result for higher order derivatives. In fact, ξ is a discontinuous function at the origin $(0,0)$; indeed,

$$\lim_{|x|\to 0} \xi(x, |x|^2) = \lim_{r \to 0^+} \frac{e^{-\frac{1}{4}}}{4\pi r^2} = \infty.$$

Consequently, the injectivity of function K implies that w has no more singularities than the points (x,t) such that

$$(x-y, K(t) - K(\tau)) = (0,0),$$

i. e., point (y,τ).

b) It is easy to prove by doing simple integral computations with the change of variable

$$x = y + (r\cos\theta, r\sin\theta). \tag{9}$$

c) For an open bounded subset Ω of $\mathbb{R}^2 \times \mathbb{R}$ we consider a set $\mathbb{R}^2 \times [a,b] \supset \Omega$. Then, from b), we have

$$\int_\Omega |w(x,t;y,\tau)|dxdt \le \int_a^b \int_{\mathbb{R}^2} w(x,t;y,\tau)dxdt = \int_a^b H(K(t)-K(\tau))dt \le b-a.$$

d) For every $s > 0$ and $\varphi \in \mathcal{D} = \mathcal{D}(\mathbb{R}^2)$ (the space of functions in $\mathcal{C}^\infty(\mathbb{R})$ with compact support), denoting by $\mathcal{D}' = \mathcal{D}'(\mathbb{R}^2)$ and using b), we have

$$\begin{aligned}\langle \xi(x-y,s), \varphi \rangle_{\mathcal{D}'\times\mathcal{D}} &= \varphi(y) \int_{\mathbb{R}^2} \xi(x-y,s)dx \\ &\quad + \int_{\mathbb{R}^2} \xi(x-y,s)(\varphi(x)-\varphi(y))dx \\ &= \varphi(y) + \int_{\mathbb{R}^2} \xi(x-y,s)(\varphi(x)-\varphi(y))dx.\end{aligned}$$

Now, if $L = ||\varphi'||_{\mathcal{C}(\mathbb{R}^2)}$, change of variable (9) and integration by parts lead to

$$\begin{aligned}\left| \int_{\mathbb{R}^2} \xi(x-y,s)(\varphi(x)-\varphi(y))dx \right| &\le \frac{L}{4\pi s} \int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{16s}} |x-y|dx \\ &= -L \int_0^\infty \left(-\frac{2r}{4s} e^{-\frac{r^2}{4s}} \right) rdr = 2L\sqrt{s} \int_0^\infty e^{-z^2} dz \\ &= L\sqrt{\pi}\sqrt{s} \to 0 \ \text{ as } \ s \to 0^+.\end{aligned}$$

e) For every $x \in \mathbb{R}^2$ and $t < \tau$

$$\frac{\partial w}{\partial t}(x,t;y,\tau) = \frac{k(t)w(x,t;y,\tau)}{K(t)-K(\tau)} \left(1 - \frac{|x-y|^2}{4(K(t)-K(\tau))} \right)$$

and, for $i = 1,2$,

$$\begin{cases} \dfrac{\partial w}{\partial x_i}(x,t;y,\tau) = -\dfrac{x_i - y_i}{2(K(t)-K(\tau))} w(x,t;y,\tau) \\ \dfrac{\partial^2 w}{\partial x_i^2}(x,t;y,\tau) = -\dfrac{w(x,t;y,\tau)}{2(K(t)-K(\tau))} \left(1 - \dfrac{(x_i-y_i)^2}{2(K(t)-K(\tau))} \right), \end{cases}$$

which concludes the result by adding and comparing the above terms suitably. □

Proposition 7 (Fundamental solution of operador $\mathcal{L}^*$) *Given $y \in B_R$ and $\tau \in (0, t_{\mathrm{f}})$, function $w : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ defined in (8) satisfies*

$$\mathcal{L}^*(w(x,t;y,\tau)) = \delta(x-y, t-\tau) \ \text{ in } \ \mathcal{D}'(\mathbb{R}^2 \times \mathbb{R}).$$

Proof. We have to prove that

$$\begin{aligned} I &\doteq \left\langle -\frac{\partial w}{\partial t}(x,t;y,\tau) - k(t)\Delta_x w(x,t;y,\tau), \varphi(x,t) \right\rangle_{\mathcal{D}'\times\mathcal{D}} \\ &= \varphi(y,\tau) \ (= \langle \delta(x-y,t-\tau), \varphi(x,t) \rangle_{\mathcal{D}'\times\mathcal{D}}),\end{aligned}$$

for every $\varphi \in \mathcal{D} = \mathcal{D}(\mathbb{R}^2 \times \mathbb{R})$, where $\mathcal{D}' = \mathcal{D}'(\mathbb{R}^2 \times \mathbb{R})$.

Since $w(x,t;y,\tau)=0$ if $t\geq\tau$, taking into account Lemma 6.c), we have

$$\begin{aligned} I &= \left\langle w(x,t;y,\tau), \frac{\partial\varphi}{\partial t}(x,t)\right\rangle_{\mathcal{D}'\times\mathcal{D}} - \langle k(t)w(x,t;y,\tau), \Delta\varphi(x,t)\rangle_{\mathcal{D}'\times\mathcal{D}} \\ &= \int_{-\infty}^{\tau}\int_{\mathbb{R}^2} w(x,t;y,\tau)\left(\frac{\partial\varphi}{\partial t}(x,t) - k(t)\Delta\varphi(x,t)\right) dxdt \\ &= \lim_{\varepsilon\to 0^+}\int_{-\infty}^{\tau-\varepsilon}\int_{\mathbb{R}^2} w(x,t;y,\tau)\left(\frac{\partial\varphi}{\partial t}(x,t) - k(t)\Delta\varphi(x,t)\right) dxdt. \end{aligned}$$

Integrating by parts, Lemma 6.e) allows to write

$$\begin{aligned} I &= \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^2}\left(w(x,t;y,\tau)\varphi(x,t)\big|_{t=-\infty}^{t=\tau-\varepsilon}\right) dx \\ &= \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^2}\xi(x-y, K(\tau-\varepsilon)-K(\tau))\varphi(x,\tau)\, dx \\ &\quad + \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^2}\xi(x-y, K(\tau-\varepsilon)-K(\tau))\left(\varphi(x,\tau-\varepsilon)-\varphi(x,\tau)\right)\, dx. \end{aligned}$$

Denoting by $L=\left\|\frac{\partial\varphi}{\partial t}\right\|_{\mathcal{C}(\mathbb{R}^3)}$, Lemma 6.b) guarantees

$$\begin{aligned} &\left|\int_{\mathbb{R}^2}\xi(x-y, K(\tau-\varepsilon)-K(\tau))\left(\varphi(x,\tau-\varepsilon)-\varphi(x,\tau)\right)\, dx\right| \\ &\qquad \leq L\int_{\mathbb{R}^2}\xi(x-y, K(\tau-\varepsilon)-K(\tau))\, dx \\ &\qquad = L\varepsilon H(K(\tau-\varepsilon)-K(\tau)) = L\varepsilon \to 0 \ \text{ as } \ \varepsilon\to 0^+. \end{aligned}$$

Finally, the result follows from Lemma 6.d). □

The fundamental solution w of operator $\mathcal{L}^*$ satisfies (5). Although w does not belong to space X, we prove that it can be approximated by functions in X satisfying equation (6). A technical result is previously stated.

Lemma 8 *Let Ω be a bounded open subset of $\mathbb{R}^N$ and $f\in\mathcal{C}^p(\Omega)\cap L^2(\Omega), p\in\mathbb{N}\cup\{0\}$. Then, for each open set ω strongly contained in Ω (this is, $\overline{\omega}\subset\Omega$), there exists a sequence $\{f_\delta\}_\delta\subset\mathcal{D}(\Omega)$ such that:*

a) $\{f_\delta\}_\delta$ strongly converges to f in $L^2(\Omega)$.

b) $\{D^\alpha f_\delta\}_\delta$ uniformly converges to $D^\alpha f$ in $\mathcal{C}(\overline{\omega})$ for $|\alpha|\leq p$.

Proof. Let $\{\hat{f}_\delta\}_\delta\subset\mathcal{D}(\Omega)$ be a sequence strongly convergent to f in $L^2(\Omega)$ (this sequence exists since $D(\Omega)$ is a dense subset of $L^2(\Omega)$; see [1]) and $\{\check{f}_\delta\}_\delta\subset\mathcal{C}^\infty(\overline{\omega})$ be a sequence satisfying that the sequences of the derivatives of order less or equal than p converge uniformly to the corresponding derivatives of f in $\overline{\omega}$ (Weierstrass Theorem).

Next, for $d=\operatorname{dist}(\overline{\omega},\partial\Omega)>0$, by choosing $\delta<\frac{d}{3}$, the δ–neighborhood ω_δ of ω (i.e., the set of points which lie at a distance from ω smaller than δ) is strongly contained in Ω. For the same choice of δ we consider f_δ satisfying

$$f_\delta = \begin{cases} \check{f}_\delta & \text{in } \overline{\omega} \\ \hat{f}_\delta & \text{in } \Omega\backslash\omega_\delta, \end{cases}$$

and defined in $\omega_\delta \backslash \overline{\omega}$ in order to be uniformly bounded in δ and such that $f_\delta \in \mathcal{C}^\infty(\Omega)$.

The sequence $\{f_\delta\}_\delta \subset \mathcal{D}(\Omega)$ satisfies b). The part a) follows from

$$\int_\Omega |f_\delta(x) - f(x)|^2 dx = \int_{\Omega\backslash\omega_\delta} |\hat{f}_\delta(x) - f(x)|^2 dx + \int_{\omega_\delta\backslash\overline{\omega}} |f_\delta(x) - f(x)|^2 dx + \int_{\overline{\omega}} |\check{f}_\delta(x) - f(x)|^2 dx$$

taking into account that the three terms on the right hand side tend to 0 as $\delta \to 0$. □

Now, we are ready to find a representation of the solution of Problem (4) in terms of its values on the boundary.

Proposition 9 *The solution of Problem (4) can be expressed, for $(y, \tau) \in B_R \times [0, t_{\mathrm{f}}]$, as*

$$\begin{aligned} v(y,\tau) = {} & \frac{h}{4\pi} \int_0^\tau \frac{f(t) - \overline{v}(R,t) - T_0}{K(t) - K(\tau)} \int_{\partial B_R} e^{-\frac{|x-y|^2}{4(K(t)-K(\tau))}} \, dxdt \\ & - \frac{1}{4\pi} \int_0^\tau \frac{k(t)\overline{v}(R,t)}{K(t) - K(\tau)} \int_{\partial B_R} \frac{\partial}{\partial \mathbf{n}} \left(e^{-\frac{|x-y|^2}{4(K(t)-K(\tau))}} \right) dxdt. \end{aligned} \tag{10}$$

Proof. For the sake of simplicity we write $w(x,t)$ (or even w) instead of $w(x,t;y,\tau)$. The key of the proof is, given $(y,\tau) \in B_R \times (0, t_{\mathrm{f}})$, to get a sequence $\{w_\varepsilon\}_\varepsilon \subset X$ of functions satisfying (5) and such that

$$\lim_{\varepsilon \to 0} \int_0^{t_{\mathrm{f}}} \int_{B_R} \mathcal{L}^*(w_\varepsilon)\, v dxdt = v(y,\tau) \tag{11}$$

and, then, to apply Corollary 4. First of all, we consider

$$\varepsilon^* = \frac{\min\{R - ||y||, \tau, t_{\mathrm{f}} - \tau\}}{2}$$

and, for $0 < \varepsilon < \varepsilon^*$, the cylinder

$$Q_\varepsilon = B_\varepsilon(y) \times (\tau - \varepsilon, \tau + \varepsilon).$$

With this choice, if $0 < \varepsilon_1 < \varepsilon_2 \le \varepsilon^*$ then $\overline{Q}_{\varepsilon_1} \subset \overline{Q}_{\varepsilon_2} \subset B_R \times (0, t_{\mathrm{f}})$. For each $\varepsilon \in (0, \varepsilon^*)$ we consider a function $\eta_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^2 \times \mathbb{R})$ satisfying

$$\eta_\varepsilon(x,t) = \begin{cases} 0, & (x,t) \in \overline{Q}_{\frac{\varepsilon}{2}} \\ \in [0,1], & (x,t) \in Q_\varepsilon \backslash \overline{Q}_{\frac{\varepsilon}{2}} \\ 1, & (x,t) \notin Q_\varepsilon \end{cases}$$

and function $w_\varepsilon(x,t) = \eta_\varepsilon(x,t) w(x,t)$. It is clear that $w_\varepsilon \in X$ and satisfies (5). Then, since w_ε is equal to w in a neighborhood of ∂B_R (and, thus, also their normal derivatives coincide), Corollary 4 implies that

$$\begin{aligned} \int_0^{t_{\mathrm{f}}} \int_{B_R} \mathcal{L}^*(w_\varepsilon)\, v dxdt = {} & h \int_0^{t_{\mathrm{f}}} (f(t) - \overline{v}(R,t) - T_0) \int_{\partial B_R} w(x,t)\, dxdt \\ & - \int_0^{t_{\mathrm{f}}} k(t)\overline{v}(R,t) \int_{\partial B_R} \frac{\partial w}{\partial \mathbf{n}}(x,t)\, dxdt. \end{aligned} \tag{12}$$

Since

$$H(K(t)-K(\tau)) = H\left(\int_t^\tau k(z)\,dz\right) = \left\{\begin{array}{l} 0 \;\text{ if }\; \tau \le t \\ 1 \;\text{ if }\; \tau > t, \end{array}\right.$$

equality (12) becomes

$$\int_0^{t_\mathrm{f}}\int_{B_R} \mathcal{L}^*(w_\varepsilon)\, v dx dt = \frac{h}{4\pi}\int_0^\tau \frac{f(t)-\overline{v}(R,t)-T_0}{K(t)-K(\tau)}\int_{\partial B_R} e^{-\frac{|x-y|^2}{4(K(t)-K(\tau))}}\,dxdt$$
$$-\frac{1}{4\pi}\int_0^\tau \frac{k(t)\overline{v}(R,t)}{K(t)-K(\tau)}\int_{\partial B_R}\frac{\partial}{\partial \mathbf{n}}\left(e^{-\frac{|x-y|^2}{4(K(t)-K(\tau))}}\right)dxdt.$$

Then, in order to obtain (10), it suffices to prove equality (11). Lemma 8 for $f = v$, $\Omega = B_R\times(0,t_\mathrm{f})$ and $\omega = Q_{\varepsilon^*}$, ensures the existence of a sequence $\{v_\delta\}_\delta \subset \mathcal{D}(B_R\times(0,t_\mathrm{f}))$ strongly convergent to v in $L^2(B_R\times(0,t_\mathrm{f}))$ and uniformly in $\overline{Q}_{\varepsilon^*}$; moreover, sequence $\{\mathcal{L}(v_\delta)\}_\delta \subset \mathcal{C}(B_R\times(0,t_\mathrm{f}))$ converges uniformly to $\mathcal{L}(v)=0$ in $\overline{Q}_{\varepsilon^*}$. In particular,

$$\lim_{\delta\to 0} v_\delta(y,\tau) = v(y,\tau) \tag{13}$$

and there exists a constant $C>0$ (independent of δ) such that

$$||\mathcal{L}(v_\delta)||_{\mathcal{C}(\overline{Q}_{\varepsilon^*})} \le C \tag{14}$$

for all $\delta > 0$. Since $\mathcal{L}^*(w_\varepsilon) \in L^2(B_R\times(0,t_\mathrm{f}))$,

$$\lim_{\varepsilon\to 0}\int_0^{t_\mathrm{f}}\int_{B_R}\mathcal{L}^*(w_\varepsilon)\, v dxdt = \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_0^{t_\mathrm{f}}\int_{B_R}\mathcal{L}^*(w_\varepsilon)\, v_\delta dxdt.$$

Thus, the proof is over if we show that

$$\lim_{(\varepsilon,\delta)\to(0,0)}\int_0^{t_\mathrm{f}}\int_{B_R}\mathcal{L}^*(w_\varepsilon)\, v_\delta dxdt = v(y,\tau),$$

For this end, for each $\sigma>0$ we look for $\delta_0 = \delta_0(\sigma) > 0$ and $\varepsilon_0 = \varepsilon_0(\sigma) > 0$ such that

$$\left|\int_0^{t_\mathrm{f}}\int_{B_R}\mathcal{L}^*(w_\varepsilon)\, v_\delta dxdt - v(y,\tau)\right| < \sigma$$

for every $0<\delta<\delta_0$ and $0<\varepsilon<\varepsilon_0$. Triangular inequality leads to

$$\left|\int_0^{t_\mathrm{f}}\int_{B_R}\mathcal{L}^*(w_\varepsilon)\, v_\delta dxdt - v(y,\tau)\right| \le \left|\int_0^{t_\mathrm{f}}\int_{B_R}\mathcal{L}^*(w_\varepsilon)\, v_\delta dxdt - v_\delta(y,\tau)\right| + \left|v_\delta(y,\tau)-v(y,\tau)\right|. \tag{15}$$

In order to estimate the first term of the right hand side of (15) we note that for every $\varphi \in \mathcal{C}(B_R\times(0,t_\mathrm{f}))$ and $0<\varepsilon<\varepsilon^*$,

$$\begin{aligned}\left|\int_0^{t_\mathrm{f}}\int_{B_R}(w_\varepsilon - w)\,\varphi\,dxdt\right| &= \left|\int_{Q_\varepsilon}(\eta_\varepsilon - 1)\,w\varphi\,dxdt\right| \le \int_{Q_\varepsilon}|w|\,|\varphi|\,dxdt \\ &\le ||\varphi||_{\mathcal{C}(\overline{Q}_{\varepsilon^*})}\int_{Q_\varepsilon}|w|\,dxdt \le 2\varepsilon||\varphi||_{\mathcal{C}(\overline{Q}_{\varepsilon^*})},\end{aligned} \tag{16}$$

where the last inequality has been obtained from Lemma 6.c). Next, choosing $\varphi = \mathcal{L}(v_\delta)$ in (16) and denoting $\mathcal{D} = \mathcal{D}(B_R \times (0, t_\mathrm{f}))$ and $\mathcal{D}' = \mathcal{D}'(B_R \times (0, t_\mathrm{f}))$, Proposition 7 allows to write

$$\left| \int_0^{t_\mathrm{f}} \int_{B_R} \mathcal{L}^*(w_\varepsilon)\, v_\delta dx dt - v_\delta(y,\tau) \right| = \left| \int_0^{t_\mathrm{f}} \int_{B_R} \mathcal{L}^*(w_\varepsilon)\, v_\delta dx dt - \langle \mathcal{L}^*(w), v_\delta \rangle_{\mathcal{D}' \times \mathcal{D}} \right|$$
$$= \left| \int_0^{t_\mathrm{f}} \int_{B_R} (w_\varepsilon - w)\, \mathcal{L}(v_\delta)\, dx dt \right| \leq \frac{2\varepsilon}{\varrho C_p} C,$$

with constant C given in (14). Hence, the choice

$$\varepsilon_0 = \min \left\{ \frac{\sigma}{4C}, \varepsilon^* \right\}$$

ensures that

$$\left| \int_0^{t_\mathrm{f}} \int_{B_R} \mathcal{L}^*(w_\varepsilon)\, v_\delta dx dt - v_\delta(y,\tau) \right| < \frac{\sigma}{2}$$

for all $\varepsilon \in (0, \varepsilon_0)$.

For the second term of the right hand side of (15), puntual convergence (13) guarantees the existence of $\delta_0 > 0$ such that

$$|v_\delta(y,\tau) - v(y,\tau)| < \frac{\sigma}{2}$$

for all $\delta \in (0, \delta_0)$.

Finally, formula (10) is also valid, in an obvious way, for $t = 0$ and extends by continuity at $t = t_\mathrm{f}$. □

Remark 10 Given $x \in \partial B_R$, function $y \to e^{-c|x-y|^2}$, defined in B_R, is not radial. Nevertheless the following function, also defined in B_R, is radial:

$$y \longmapsto \int_{\partial B_R} e^{-c|x-y|^2}\, dx.$$

Therefore, the function given in (10) is radial (although it does not look so).

Next, we write the solution of Problem (4) in polar coordinates:

Corollary 11 *Denoting by*

$$\gamma(r,\theta) = R^2 - 2Rr\cos\theta + r^2 \quad \textit{and} \quad g(t,\tau) = \frac{1}{K(\tau) - K(t)},$$

the solution of Problem (4) satisfies:

$$\overline{v}(r,t) = \frac{Rh}{4\pi} \int_0^t (f(\tau) - \overline{v}(R,\tau) - T_0) g(t,\tau) \int_0^{2\pi} e^{-\frac{\gamma(r,\theta) g(t,\tau)}{4}}\, d\theta d\tau$$
$$+ \frac{R}{2\pi} \int_0^t \frac{\partial \overline{v}}{\partial \tau}(R,\tau) \int_0^{2\pi} \frac{R - r\cos\theta}{\gamma(r,\theta)} e^{-\frac{\gamma(r,\theta) g(t,\tau)}{4}}\, d\theta d\tau$$

for $r \in [0, R)$ *and* $t \in [0, t_\mathrm{f}]$.

Proof. For $(r,t)\in[0,R)\times[0,t_{\mathrm{f}}]$ given, we choose $x=(r,0)\in\mathbb{R}^2$. By changing the roles of t with τ and x with y, from Proposition 9 we have

$$v(x,t)=\frac{h}{4\pi}\int_0^t (f(\tau)-\overline{v}(R,\tau)-T_0)g(t,\tau)\int_{\partial B_R} e^{-\frac{|x-y|^2 g(t,\tau)}{4}}\,dyd\tau$$
$$-\frac{1}{4\pi}\int_0^t k(\tau)\overline{v}(R,\tau)g(t,\tau)\int_{\partial B_R}\frac{\partial}{\partial\mathbf{n}}\left(e^{-\frac{|x-y|^2 g(t,\tau)}{4}}\right)dyd\tau.$$

Then, for all $y=(R\cos\theta,R\sin\theta)\in\partial B_{\mathrm{R}}$,

$$|x-y|^2=R^2-2Rr\cos\theta+r^2=\gamma(r,\theta),$$

which leads to

$$\int_{\partial B_R} e^{-\frac{|x-y|^2 g(t,\tau)}{4}}\,dy=R\int_0^{2\pi} e^{-\frac{\gamma(r,\theta)g(t,\tau)}{4}}\,d\theta$$

by means of the change of variable $y=(R\cos\theta,R\sin\theta)$. Moreover,

$$\int_{\partial B_R}\frac{\partial}{\partial\mathbf{n}}\left(e^{-\frac{|x-y|^2 g(t,\tau)}{4}}\right)dy=R\int_0^{2\pi}\frac{\partial}{\partial\rho}\left(e^{-\frac{(\rho^2-2\rho r\cos\theta+r^2)g(t,\tau)}{4}}\right)\Big|_{\rho=R}d\theta$$
$$=-\frac{R}{2}\int_0^{2\pi}(R-r\cos\theta)g(t,\tau)e^{-\frac{\gamma(r,\theta)g(t,\tau)}{4}}\,d\theta.$$

Now, by taking into account that

$$\frac{\partial g}{\partial\tau}(t,\tau)=-\frac{K'(\tau)}{(K(\tau)-K(t))^2}=k(\tau)g^2(t,\tau)$$

we can write

$$\overline{v}(r,t)=\frac{Rh}{4\pi}\int_0^t (f(\tau)-\overline{v}(R,\tau)-T_0)g(t,\tau)\int_0^{2\pi} e^{-\frac{\gamma(r,\theta)g(t,\tau)}{4}}\,d\theta d\tau$$
$$+\frac{R}{8\pi}\int_0^t \overline{v}(R,\tau)k(\tau)g^2(t,\tau)\int_0^{2\pi}(R-r\cos\theta)e^{-\frac{\gamma(r,\theta)g(t,\tau)}{4}}\,d\theta d\tau$$
$$=\frac{Rh}{4\pi}\int_0^t (f(\tau)-\overline{v}(R,\tau)-T_0)g(t,\tau)\int_0^{2\pi} e^{-\frac{\gamma(r,\theta)g(t,\tau)}{4}}\,d\theta d\tau$$
$$-\frac{R}{2\pi}\int_0^{2\pi}\frac{R-r\cos\theta}{\gamma(r,\theta)}\int_0^t \overline{v}(R,\tau)\frac{\partial}{\partial\tau}(e^{-\frac{\gamma(r,\theta)g(t,\tau)}{4}})\,d\tau d\theta.$$

The result holds integrating by parts with respect to τ in the second term of the right hand side of the last expression. □

The main result of this section, related to Problem (1), can be stated as follows:

Theorem 12 *Denoting by*

$$m(t)=\overline{T}(R,t)\quad and\quad Q(t,\tau)=e^{\alpha(P(t)-P(\tau))},$$

the solution of Problem (1) can be expressed as

$$\overline{T}(r,t)=T_0Q(t,0)+\frac{Rh}{4\pi}\int_0^t (T^e(\tau)-m(\tau))\,Q(t,\tau)g(t,\tau)\int_0^{2\pi} e^{-\frac{\gamma(r,\theta)g(t,\tau)}{4}}\,d\theta d\tau$$
$$+\frac{R}{2\pi}\int_0^t (m'(\tau)-\alpha m(\tau)P'(\tau))\,Q(t,\tau)\int_0^{2\pi}\frac{R-r\cos\theta}{\gamma(r,\theta)}\,e^{-\frac{\gamma(r,\theta)g(t,\tau)}{4}}\,d\theta d\tau$$

for $r\in[0,R)$ and $t\in[0,t_{\mathrm{f}}]$.

Proof. From Corollary 11, undoing the change of variable (3), we obtain

$$\overline{u}(r,t) = T_0 + \overline{v}(r,t) = T_0 + \frac{Rh}{4\pi}\int_0^t (f(\tau) - \overline{u}(R,\tau))g(t,\tau)\int_0^{2\pi} e^{-\frac{\gamma(r,\theta)g(t,\tau)}{4}}\, d\theta d\tau$$
$$+\frac{R}{2\pi}\int_0^t \frac{\partial \overline{u}}{\partial \tau}(R,\tau)\int_0^{2\pi}\frac{R - r\cos\theta}{\gamma(r,\theta)}\, e^{-\frac{\gamma(r,\theta)g(t,\tau)}{4}}\, d\theta d\tau.$$

The result follows undoing the change of variable (2), taking into account that

$$\frac{\partial \overline{u}}{\partial \tau}(R,\tau) = \frac{\partial}{\partial \tau}(m(\tau)e^{\alpha(P(0)-P(\tau))}) = (m'(\tau) - P'(\tau)\alpha m(\tau))Q(0,\tau)$$

and

$$Q(t,0)Q(0,\tau) = Q(t,\tau).$$

□

Remark 13 Although function g satisfies

$$\lim_{\tau\to t} g(t,\tau) = \lim_{\tau\to t}\frac{1}{K(\tau) - K(t)} = \infty,$$

the integrands in Theorem 12 are well defined; more precisely, since $r < R$ then $\gamma(r,\theta) \neq 0$ and, therefore, both integrands vanish in $\tau = t$.

4. Uniqueness of solution for the inverse problem

In this section we deal with the uniqueness of solution for the inverse problem, and it is proved that, under suitable assumptions, function k is uniquely determined by the values that function $\overline{T}$ takes at R and at any other point $r_0 \in [0,R)$.

It is noteworthy that this does not always happen. For example, when the ambient temperature evolves according to

$$T^e(t) = T_0 e^{\alpha(P(t)-P(0))},$$

$T(t) = T^e(t)$ itself is a solution of the direct problem (1) for any function k.

In order to get the uniqueness of solution of the problem of identifying the thermal diffusivity, we work under the following assumptions:

(H1) $T^e(t) \equiv T_0$ for all $t \in [0, t_\mathrm{f}]$.

(H2) P is a linear function in $[0, t_\mathrm{f}]$ with $P' \equiv \beta > 0$.

Remark 14 These assumptions do not impose any restriction on the problem of identifying k, but on the experiment in which measurements are obtained.

Remark 15 Under assumptions **(H1)** and **(H2)**, the solution of Problem (1) can be written as

$$\overline{T}(r,t) = T_0 Q(t,0) + \frac{Rh}{4\pi}\int_0^t (T_0 - m(\tau))\, Q(t,\tau)g(t,\tau)\int_0^{2\pi} e^{-\frac{\gamma(r,\theta)g(t,\tau)}{4}}\, d\theta d\tau$$
$$+\frac{R}{2\pi}\int_0^t (m'(\tau) - \alpha\beta m(\tau))\, Q(t,\tau)\int_0^{2\pi}\frac{R - r\cos\theta}{\gamma(r,\theta)}\, e^{-\frac{\gamma(r,\theta)g(t,\tau)}{4}}\, d\theta d\tau$$

(see Theorem 12).

Comparison Principle allows to prove the following auxiliary results:

Lemma 16 *Under assumptions* **(H1)** *and* **(H2)***, if k is a Lipschitz continuous function in $[0, t_\mathrm{f}]$ with $k \geq k_0 > 0$, then*

$$T_0 \leq T(x,t) \leq T_0 e^{\alpha(P(t)-P(0))} \tag{17}$$

for every $(x,t) \in \overline{B_R} \times [0, t_\mathrm{f}]$. Moreover, for each $t^* \in (0, t_\mathrm{f})$ there exists $\tau^* \in (0, t^*)$ such that $T_0 < m(\tau^*)$.

Proof. By using changes of variable (2) and (3), inequality (17) becomes

$$f(t) - T_0 \leq v(x,t) \leq 0, \tag{18}$$

which arises directly from Comparison Principle in Proposition 5.

In order to prove the final state, we suppose that there exists an interval $(0, t^*)$, with $0 < t^* < t_\mathrm{f}$, and such that $m(t) = T_0$ for all $t \in (0, t^*)$. Thus, function T solves the problem

$$\begin{cases} \dfrac{\partial T}{\partial t} - k(t)\Delta T = \alpha P'(t) T & \text{in } B_R \times (0, t^*) \\ k(t)\dfrac{\partial T}{\partial \mathbf{n}} = 0 & \text{on } \partial B_R \times (0, t^*) \\ T = T_0 & \text{on } B_R \times \{0\}. \end{cases}$$

Uniqueness of solution of this problem (see, e. g., [14, Theorem 8]) forces T to coincide on $\partial B_R \times (0, t^*)$ with the (unique) solution of this problem $T_0 e^{\alpha(P(t)-P(0))}$, which satisfies $m(t) \neq T_0$ for all $t > 0$. □

Lemma 17 *Under assumptions* **(H1)** *and* **(H2)***, if $k \in \mathcal{C}^1([0, t_\mathrm{f}])$, $k \geq k_0 > 0$ and*

$$k'(t) \leq k(t)\, \frac{f'(t)}{f(t) - T_0} = \alpha\beta \frac{k(t)}{e^{\alpha\beta t} - 1} \tag{19}$$

for $t \in [0, t_\mathrm{f}]$, then

$$\frac{\partial T}{\partial t}(x,t) - \alpha\beta T(x,t) \leq 0 \tag{20}$$

for every $(x,t) \in \overline{B_R} \times [0, t_\mathrm{f}]$. In particular,

$$m'(t) - \alpha\beta m(t) \leq 0$$

for $t \in (0, t_\mathrm{f})$.

Proof. Again, using changes of variable (2) and (3), inequality (20) can be rewritten as

$$\frac{\partial v}{\partial t}(x,t) \leq 0.$$

From (4), $\dfrac{\partial v}{\partial t}$ satisfies

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial t}\right) - k\Delta\left(\frac{\partial v}{\partial t}\right) = k'\Delta v = \frac{k'}{k}\,(k\Delta v) = \frac{k'}{k}\frac{\partial v}{\partial t}$$

and

$$k\frac{\partial}{\partial \mathbf{n}}\left(\frac{\partial v}{\partial t}\right)+h\left(\frac{\partial v}{\partial t}\right)=hf'-k'\frac{\partial v}{\partial \mathbf{n}}=hf'-\frac{k'}{k}h(f-T_0-v).$$

Furthermore, since $v\equiv 0$ at $t=0$, from (4)

$$\frac{\partial v}{\partial t}(x,0)=0.$$

Then $\mathcal{V}=\dfrac{\partial v}{\partial t}$ is a solution of

$$\begin{cases} \dfrac{\partial \mathcal{V}}{\partial t}-k\Delta\mathcal{V}=\dfrac{k'}{k}\mathcal{V} & \text{in } B_R\times(0,t_{\mathrm{f}})\\ k\dfrac{\partial \mathcal{V}}{\partial \mathbf{n}}+h\mathcal{V}=h\left(f'-\dfrac{k'}{k}(f-T_0-v)\right) & \text{on } \partial B_R\times(0,t_{\mathrm{f}})\\ \mathcal{V}=0 & \text{on } B_R\times\{0\}.\end{cases}$$

Therefore, if $\zeta=\dfrac{1}{k}\mathcal{V}=\dfrac{1}{k}\dfrac{\partial v}{\partial t}$, it satisfies

$$\begin{cases} \dfrac{\partial \zeta}{\partial t}-k\Delta\zeta=0 & \text{in } B_R\times(0,t_{\mathrm{f}})\\ k\dfrac{\partial \zeta}{\partial \mathbf{n}}+h\zeta=\dfrac{h}{k}\left(f'-\dfrac{k'}{k}(f-T_0-v)\right) & \text{on } \partial B_R\times(0,t_{\mathrm{f}})\\ \zeta=0 & \text{on } B_R\times\{0\}.\end{cases}$$

Inequality (19) will allow to prove that

$$\frac{h}{k}\left(f'-\frac{k'}{k}(f-T_0-v)\right)\le 0,\ (x,t)\in\partial B_R\times(0,t_{\mathrm{f}}). \tag{21}$$

With that aim we distinguish two cases: times t such that $k'(t)\le 0$ and times t such that $k'(t)>0$.

1) If $k'(t)\le 0$: From (18) $f(t)-T_0-v(x,t)\le 0$. Inequality (21) is obtained by using that

$$f'(t)=-\alpha\beta T_0\, e^{-\alpha\beta t}\le 0.$$

2) If $k'(t)>0$: From the last inequality in (18),

$$\begin{aligned}\frac{h}{k(t)}\left(f'(t)-\frac{k'(t)}{k(t)}(f(t)-T_0-v(x,t))\right) &= \frac{h}{k(t)}\left(f'(t)-\frac{k'(t)}{k(t)}(f(t)-T_0)\right)\\ &\quad+\frac{hk'(t)}{k^2(t)}v(x,t)\\ &\le \frac{h}{k^2(t)}(k(t)f'(t)-k'(t)(f(t)-T_0)).\end{aligned}$$

Inequality (19) and $f(t)-T_0\le 0$ provide the required sign.

Once inequality (21) has been proven, the Comparison Principle showed in Proposition 5 allows to conclude the result. □

Remark 18 Lemma 17 is also true if $k \in W^{1,\infty}(0,t_\mathrm{f})$ (cf. [1, Proposition IX.4, pag. 155]).

Remark 19 Assumption (19) on the growth of k supposes only a restriction on the time intervals where k is an increasing function (in fact, it is automatically fulfilled by constant or decreasing functions). In addition, since

$$\lim_{t\to 0^+} \frac{1}{e^{\alpha\beta t}-1} = +\infty,$$

such a condition does not suppose any restriction in k for short times. It is *a priori* information needed to identify the conductivity coefficient. This information means that the coefficient k can not have abrupt changes, typical of the processes that produce phase change, which does not occur in the cases motivating this work, as discussed in the Introduction.

Let us suppose that there exist two different functions k_1 and k_2 which provide the same measurement $m(t)$ at the right end R and, moreover, the same measurement at some interior point $r_0 \in [0,R)$.

We assume the following property for k_1 and k_2: there exist $t_0 \in [0,t_\mathrm{f}]$ and $t^* \in (t_0,t_\mathrm{f}]$ such that

$$\begin{cases} k_1(t) = k_2(t), & t\in[0,t_0] \\ k_1(t) > k_2(t), & t \in (t_0,t^*]. \end{cases}$$

This implies that the above functions can not have oscillations as $t\sin\left(\frac{1}{t}\right)$ around $t=0$. A sufficient condition for this is that k_1 and k_2 be continuous and right locally analytic functions.

Let us denote

$$\gamma_0(\theta) = \gamma(r_0,\theta),\ \psi_0(\theta) = \frac{R - r_0\cos\theta}{\gamma_0(\theta)},$$

$$K_i(t) = \int_t^{t_\mathrm{f}} k_i(s)ds \ \text{ and } \ x_i(\tau) = \frac{1}{K_i(\tau)-K_i(t^*)} = \frac{1}{\displaystyle\int_\tau^{t^*} k_i(s)ds},\ i=1,2.$$

Since measurements at $r=r_0$ for $k_1(t)$ and $k_2(t)$ are the same at any time, the expression of the solution in Remark 15 at $t=t^*$ leads to

$$\begin{aligned} &T_0Q(t^*,0) + \frac{Rh}{4\pi}\int_0^{t^*}\left(T_0 - m(\tau)\right)Q(t^*,\tau)x_1(\tau)\int_0^{2\pi} e^{-\frac{\gamma_0(\theta)x_1(\tau)}{4}}\,d\theta d\tau \\ &+ \frac{R}{2\pi}\int_0^{t^*}\left(m'(\tau)-\alpha\beta m(\tau)\right)Q(t^*,\tau)\int_0^{2\pi}\psi_0(\theta)\,e^{-\frac{\gamma_0(\theta)x_1(\tau)}{4}}\,d\theta d\tau \\ =\ &T_0Q(t^*,0) + \frac{Rh}{4\pi}\int_0^{t^*}\left(T_0 - m(\tau)\right)Q(t^*,\tau)x_2(\tau)\int_0^{2\pi} e^{-\frac{\gamma_0(\theta)x_2(\tau)}{4}}\,d\theta d\tau \\ &+ \frac{R}{2\pi}\int_0^{t^*}\left(m'(\tau)-\alpha\beta m(\tau)\right)Q(t^*,\tau)\int_0^{2\pi}\psi_0(\theta)\,e^{-\frac{\gamma_0(\theta)x_2(\tau)}{4}}\,d\theta d\tau. \end{aligned}$$

Consequently,

$$0 = \frac{Rh}{4\pi}\int_0^{t^*}(T_0 - m(\tau))\,Q(t^*,\tau)\int_0^{2\pi}\left(x_1(\tau)e^{-\frac{\gamma_0(\theta)x_1(\tau)}{4}} - x_2(\tau)e^{-\frac{\gamma_0(\theta)x_2(\tau)}{4}}\right)d\theta d\tau \tag{22}$$

$$+\frac{R}{2\pi}\int_0^{t^*}(m'(\tau) - \alpha\beta m(\tau))\,Q(t^*,\tau)\int_0^{2\pi}\psi_0(\theta)\left(e^{-\frac{\gamma_0(\theta)x_1(\tau)}{4}} - e^{-\frac{\gamma_0(\theta)x_2(\tau)}{4}}\right)d\theta d\tau. \tag{23}$$

We point out that $x_1(\tau) < x_2(\tau)$ for $0 < \tau < t^*$. Moreover, since xe^{-cx} is a strictly decreasing function for $x \geq \frac{1}{c}$, for the choice

$$c = \frac{\gamma_0(\theta)}{4} = \frac{R^2 - 2Rr_0\cos\theta + r_0^2}{4}, \tag{24}$$

we have that

$$x_1(\tau)e^{-cx_1(\tau)} - x_2(\tau)e^{-cx_2(\tau)} > 0$$

whenever $x_1 \geq \frac{1}{c}$. This condition is fulfilled if functions k_i satisfy, for example,

$$\int_0^{t_f} k_i(s)ds \leq \frac{(R - r_0)^2}{4} \leq c,\ i = 1, 2,$$

which should be interpreted as part of the a priori information that has to be known in order to determine k uniquely.

Furthermore, since e^{-cx} with c given in (24) is a decreasing function, we have that

$$e^{-cx_1(\tau)} - e^{-cx_2(\tau)} > 0.$$

Now, hypothesis **(H1)** and **(H2)**, and Lemmas 16 and 17, lead (22)–(23) to a null sum of two definite integrals of non positive functions. Therefore, since the integrand in (22) takes values strictly negative (in a neighborhood of point τ^* of Lemma 16), we arrive to a contradiction. Consequently, it is not possible (as we had assumed) to find a point $t^* \in [0, t_f]$ so that $k_1(t^*) \neq k_2(t^*)$.

Collecting the above reasonings, the following result is proved:

Theorem 20 *Let T_1 and T_2 be, respectively, the solutions of problems*

$$\begin{cases} \varrho C_p\dfrac{\partial T}{\partial t} - k_1(t)\Delta T = \alpha P'(t)T & \text{in } B_R \times (0, t_f) \\ k_1(t)\dfrac{\partial T}{\partial \mathbf{n}} = h\,(T^e(t) - T) & \text{on } \partial B_R \times (0, t_f) \\ T = T_0 & \text{on } B_R \times \{0\} \end{cases}$$

and

$$\begin{cases} \varrho C_p\dfrac{\partial T}{\partial t} - k_2(t)\Delta T = \alpha P'(t)T & \text{in } B_R \times (0, t_f) \\ k_2(t)\dfrac{\partial T}{\partial \mathbf{n}} = h\,(T^e(t) - T) & \text{on } \partial B_R \times (0, t_f) \\ T = T_0 & \text{on } B_R \times \{0\}, \end{cases}$$

where $k_i \in \mathcal{C}^1([0,t_\mathrm{f}])$ *and* $k_i \geq k_0 > 0$ *for* $i = 1, 2$. *Let us suppose* **(H1)**, **(H2)** *and*

$$\overline{T}_1(R,t) = \overline{T}_2(R,t) \ \ \textit{and} \ \ \overline{T}_1(r_0,t) = \overline{T}_2(r_0,t) \ \ \textit{for some} \ \ r_0 \in [0,R)$$

hold for all $t \in [0, t_\mathrm{f}]$.

Assuming k_i *are right locally analytic functions in* $[0, t_\mathrm{f})$,

$$\int_0^{t_\mathrm{f}} k_i(s)ds \leq \frac{(R-r_0)^2}{4}, \ i = 1, 2 \tag{25}$$

and

$$k_i'(t) \leq \alpha\beta\frac{k_i(t)}{e^{\alpha\beta t}-1}, \ t \in [0, t_\mathrm{f}], \ i = 1, 2, \tag{26}$$

then $k_1 = k_2$.

Remark 21 The further the two measurements are (i.e., the closer to zero is r_0) the less restrictive is condition (25) on the a priori information on k and, therefore, the uniqueness of the inverse problem solution can be guaranteed for a wider set of functions.

Remark 22 In the particular case where k is a constant, it is possible to prove the uniqueness result of Theorem 20 without requiring any assumptions about the size or growth of k (i.e. without requiring assumptions (25) and (26)).

5. Conclusions

The inverse problem considered in this work has an extra difficulty with respect to those studied in other works found in the literature. We have a heat transfer system with a Robin type boundary condition including the time dependent variable that we want to identify. If one tries to reduce the problem to an operational problem, an abstract parabolic equation would be obtained with a time dependent operator. This complicates the obtention of an expression for the solution, either in integral or in series form. In the works found in the literature for problems similar to this one (see Section 1) these kind of expressions are fundamental when using the overdetermination condition in order to solve the identification problem. That is why in the problem considered in this work we need two overdetermination conditions, instead of one, to show the uniqueness property in the identification problem considered.

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